Critical behaviour of the 1D q-state Potts model with long-range interactions

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Abstract. The critical behaviour of the one-dimensional q-state Potts model with long-range interactions decaying with distance r as $r^{-(1+\sigma)}$ has been studied in the wide range of parameters $0 < \sigma \le 1$ and $\frac{1}{16} \le q \le 64$. A transfer matrix has been constructed for a truncated range of interactions for integer and continuous q, and finite range scaling has been applied. Results for the phase diagram and the correlation length critical exponent are presented.

1. Introduction

Recently, finite range scaling (FRS) appeared as an efficient method to deal with various discrete one-dimensional models with long range (LR) interactions (Glumac and Uzelac 1989, 1991). The aim of the present study is to apply it to the one-dimensional q-state Potts model with LR interactions, defined by the Hamiltonian

$$-\beta H = K \sum_{i<j} \frac{1}{(j-i)^{1+\sigma}} \left[\delta(s_i, s_j) - \frac{1}{q} \right] \tag{1}$$

where the interaction constant is chosen to be equal to k_B and the inverse temperature is denoted by K. The symbol s_i denotes a Potts variable at site i which can assume q different states, while $\delta(s_i, s_j)$ is the Kronecker symbol.

Such a model is expected to have, due to the LR of interactions, a non-trivial phase transition at finite temperature, like an Ising model (as its special case $q = 2$), although the problem becomes much more complex due to the possibility of changing degrees of freedom through parameter q.

The Potts model has been extensively studied in the case of short range (SR) interactions. In the two-dimensional SR case for example, besides integer-q models which are equivalent to the classical $(q-1)/2$-spin models, there are also non-integer q models which are connected with other critical problems such as percolation ($q = 1$ limit), resistor network ($q = 0$ limit), dilute spin glass ($q = 1/2$ limit), branched polymers ($0 \le q \le 1$) (see review by Wu 1982).

It would be interesting to extend those concepts to LR interactions. However, in contrast to the rich literature on the SR Potts model, the LR Potts model has been much less explored. The best explored member of the q-state Potts models is the $q = 2$ model where both analytical (Dyson 1969, Fisher *et al* 1972, Kosterlitz 1976) and numerical (Nagle and Bonner 1970, Glumac and Uzelac 1989, 1991) results are available. Dyson has shown that, as a consequence of the LR forces, there is a critical temperature different from zero as long as $0 < \sigma \le 1$. Fisher *et al* (1972) have performed $\varepsilon = 2\sigma - d$

and $1/n$ expansions in the context of the renormalization group (RG) for all dimensions d and $\sigma \neq 2$ on a more general system with an n-component order parameter. They obtained the correlation function exponent $\eta = 2 - \sigma$ to all orders in ε, the correlation length exponent $\nu = 1/\sigma$ when $\varepsilon < 0$ (classical regime), $\nu = 1/\sigma + [(n+2)/(n+8)](\varepsilon/\sigma^2) + \ldots$ when $\varepsilon > 0$, and leading irrelevant exponent $y_3 = -|\varepsilon|$. A similar expansion, in $1 - \sigma > 0$, has been done by Kosterlitz (1976), who obtained: $1/\nu = -y_3 = [2(1-\sigma)]^{1/2}$ when $\sigma \to 1$. Very recently, a new method, called cycle expansion, has been demonstrated by Mainieri (1992) on the example of the $s = \frac{1}{2}$ Ising model. Mainieri obtained very accurate estimates of the critical temperature and other thermodynamic quantities in the critical region.

The literature on $q \neq 2$ LR Potts models is relatively poor. By using the RG ε-expansion, Priest and Lubensky (1976) have found a fixed point and critical exponent ν to the first order in $\varepsilon = 3\sigma - d$. More recently, Theumann and Gusmão (1985), performed a similar calculation to the second order of $\varepsilon = 3\sigma - d$. They obtained the following results for $q < 3$: $\eta = 2 - \sigma$ to all orders of ε, and

$$\nu^{-1} = \sigma - \frac{\alpha}{2\beta}\varepsilon + \frac{\alpha\beta(\alpha-\beta)S(\sigma) + \frac{1}{2}\alpha(\beta^2-\gamma)G(\sigma)}{8\beta^3}\varepsilon^2 + 0(\varepsilon^3) \tag{2}$$

where $\alpha = q^2(q-2)$, $\beta = q^2(q-3)$, $\gamma = q^4(q^2 - 6q + 10)$,

$$S(\sigma) \equiv \psi(\tfrac{3}{2}\sigma) - \psi(\sigma) - \psi(\sigma/2) + \psi(1)$$

and $G(\sigma) \equiv [\Gamma(\tfrac{3}{2}\sigma)\Gamma^3(\sigma/2)]/\Gamma^3(\sigma)$. $\Psi(z)$ is the logarithmic derivative of the Γ function. In the $q \to 1$ limit, their results are applicable to the description of percolation critical behaviour for a random Ising ferromagnet with a LR power-law interaction.

The advantage of the FRS method is the capability to cover, unlike the ε-expansion, a wide range of parameter space. In the present article the phase diagram and the correlation length critical exponent will be studied in the region $0 < \sigma \leqslant 1$ and $\frac{1}{16} \leqslant q \leqslant 64$.

The main idea of our approach is the following: we truncate the originally infinite-range interaction to the L first neighbours and solve exactly (although numerically) the finite-range version of (1)

$$-\beta H = \sum_{i=1}^{N} \sum_{j=1}^{L} K_j \left[\delta(s_i, s_{i+j}) - \frac{1}{q} \right]_{pbc} \tag{3}$$

where K_j denotes $K/j^{1+\sigma}$. Then, by using the FRS method (section 2.1) with the appropriate extrapolating procedure (section 3), the $L \to \infty$ behaviour will be deduced.

For this purpose a transfer matrix for a reduced model defined by equation (3) will be constructed. Two different formalisms will be applied: one for integer q values (section 2.2) and the other for non-integer q values (section 2.3).

The results are presented in section 3, while section 4 contains a short discussion and some open questions.

2. Method

2.1. FRS

The FRS method has been constructed in analogy with the FSS (Fisher and Barber 1972), where instead of finite-size, the finite-range of interactions is scaled (Glumac

and Uzelac 1989, 1991). The basic idea of FRS is that by studying the sequence of systems with their long-range interactions truncated above certain range, one can obtain, by using scaling properties, the information on the critical behaviour of the true infinite system.

Let $A_\infty(t)$ be the physical quantity of the true long-range system, which algebraically diverges, in the vicinity of the critical point $t = 0$

$$A_\infty(t) \simeq A_0 t^{-\rho} \tag{4}$$

where $t = (T - T_c)/T_c$, T_c is the critical temperature, ρ is the related critical exponent, and A_0 is a constant. Then, analogous to the FSS hypothesis one can assume that for large finite range M and small t, $A_M(t)$ can be written in the form

$$A_M(t) = A_\infty(t) f\left(\frac{M}{\xi_\infty}\right) \tag{5}$$

where f is a homogeneous function with following properties

$$\lim_{x \to \infty} f(x) = 1 \qquad \lim_{x \to 0} f(x) = \text{constant.} \; x^{\rho/\nu}. \tag{6}$$

By applying equation (5) to the correlation length $\xi_\infty(t) = \xi_0 t^{-\nu}$, the standard scaling procedure gives the condition for critical temperature through the fixed point equation

$$\frac{\xi_M(t^*)}{M} = \frac{\xi_{M'}(t^*)}{M'} \tag{7}$$

and the expression for correlation length critical exponent ν

$$\nu^{-1} = \frac{\ln \dfrac{\xi'_M(t^*)}{\xi'_{M'}(t^*)}}{\ln \dfrac{M}{M'}} - 1 \tag{8}$$

where $M' = M - 1$ in all calculations, and ξ' is temperature derivative of ξ.

There are two important facts concerning the above method which are of interest. First, since the critical behaviour is essentially dependent on the range of interaction, one can (unlike in the FSS (Brézin 1982)) expect applicability of FRS in mean-field as well as in the non-trivial region (Glumac and Uzelac 1989, 1991). Second, the correlation length calculated from the transfer matrix presents the average distance between domain walls for an infinitely long strip so that relation (7) gives the transition temperature for both first and second-order phase transitions. This fact is common with FSS (Binder 1987).

2.2. Transfer matrix: integer-q formalism

It is straightforward to construct the transfer matrix for model (3) with an integer number of Potts states.

The chain with a range of interaction L, can be represented as a strip with columns of height L (Uzelac and Glumac 1988). Each column then can be considered as an object with q^L possible states, interacting only with its first neighbour.

The transfer matrix T is given by:

$$\langle i|T|j \rangle = \exp\left\{ \sum_{k=1}^{L} K_k \left[\sum_{n=1}^{L-k} \delta(i_n, i_{n+k}) + \sum_{n=1}^{k} \delta(i_{L+n-k}, j_n) \right] \right\} \tag{9}$$

where $j_m = 0, 1, \ldots, q-1$ are the elements of the L-component vector of states $|j\rangle$ of a column of height L. The matrix T can be further decomposed into a product of L matrices T_n, each one describing the addition of one more site to the column (Temperley and Lieb 1971).

$$T = T_1 \cdot \ldots \cdot T_L \tag{10}$$

$$\langle i|T_n|j\rangle = \prod_{\substack{l=1 \\ l \neq n}}^{L} \delta(i_l, j_l) \exp\left\{ \sum_{k=1}^{n-1} K_k \delta(j_{n-k}, j_n) + \sum_{k=n}^{L-1} K_k \delta(j_{L+n-k}, j_n) + K_L \delta(i_n, j_n) \right\}. \tag{11}$$

There is a simple relation between neighbouring one-site matrices:

$$U^T T_{n+1} U = T_n \tag{12}$$

where $T_{L+1} = T_1$ and U is a matrix of the translation operator in the vertical strip direction, with matrix elements given by

$$\langle i|U|j\rangle = \delta(i_1, j_L)\delta(i_2, j_1)\delta(i_3, j_2), \ldots, \delta(i_{L-1}, j_{L-2})\delta(i_L, j_{L-1}). \tag{13}$$

The matrix U satisfies the relations $U^L = 1$ and $U^{L-1} = U^T = U^{-1}$. Consequently, the transfer matrix can be written as the Lth power of a single matrix as a peculiarity of the present long-range model:

$$T = (U \cdot T_L)^L = \tilde{T}^L \tag{14}$$

$$\langle i|\tilde{T}|j\rangle = \delta(j_1, i_2)\delta(j_2, i_3) \ldots \delta(j_{L-1}, i_L) \exp\left\{ \sum_{m=1}^{L} K_{L+1-m} \delta(j_L, i_m) \right\}. \tag{15}$$

Notice that $\tilde{T}$ has only q non-zero elements per row which greatly reduces computer memory.

Further reduction is obtained by using the symmetry properties of Potts interaction. In the present calculation, only the translation invariance in the space of Potts states has been used, which decomposes $\tilde{T}$ into q submatrices (of order q^{L-1}). $q-1$ of them have identical eigenvalues as a consequence of Kronecker-δ type interaction. Thus we have to diagonalize only two submatrices or, more precisely, we have to find only the largest eigenvalue of each of the two submatrices, $\mu_{1,L}$ and $\mu_{2,L}$ respectively, to be able to construct the correlation length ξ_L

$$\xi_L = \frac{L}{\ln \lambda_{1,L}/\lambda_{2,L}} = \frac{1}{\ln \mu_{1,L}/\mu_{2,L}} \tag{16}$$

where $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2$ are the eigenvalues of T and $\tilde{T}$ respectively.

For this purpose, the method of direct iteration with a single vector is suitable (Wilkinson 1965). If one defines a set of vectors b_j in the following way.

$$Ab_j = b_{j+1} \tag{17}$$

for arbitrary initial value b_0, where A is a real non-symmetric matrix, then the largest eigenvalue μ of matrix A is given by

$$\mu = \lim_{j \to \infty} \frac{b_j(n)}{b_{j-1}(n)} \tag{18}$$

for any vector's component n. By applying this method for cases with $q = 2, 3, 4$ and 5, we obtained the two largest eigenvalues of $\tilde{T}$ for maximum ranges $L = 20, 13, 10$

and 9, respectively. A few tens to a few hundred iterations were required to obtain the leading eigenvalue with an accuracy of 10^{-15} for both submatrices.

2.3. Transfer matrix: continuous-q formalism

As mentioned in the introduction, there are some interesting cases ($q = 0, 1/2, 1, \ldots$) to which the described integer-q formalism cannot be applied or becomes inefficient (limitation to small ranges for large q models). Thus, it would be advantageous to formulate a q-independent transfer matrix, which would allow treatment of q as a continuous parameter. Such a type of transfer matrix has already been constructed for the 2D Potts model with SR interaction (Blöte *et al* 1981, Blöte and Nightingale 1982). We will shortly explain the application of that method to the considered *LR* model.

Let us translate the partition function of model (3), into the graph-theory language. First, we rewrite the basic expression $\exp\{K_j \cdot \delta(s_i, s_{i+j})\}$ as

$$e^{K_j \delta(s_i, s_{i+j})} = 1 + v_j \delta(s_i, s_{i+j}) \tag{19}$$

where $v_j = e^{K_j} - 1$, so that the partition function, with periodic boundary conditions, becomes

$$Z_{N,L} = \sum_{s_1=1}^{q} \sum_{s_2=1}^{q} \ldots \sum_{s_N=1}^{q} \prod_{j=1}^{L} [1 + v_j \delta(s_1, s_{1+j})][1 + v_j \delta(s_2, s_{2+j})] \ldots [1 + v_j \delta(s_N, s_{N+j})]. \tag{20}$$

If we perform the multiplications of the above terms, the one-to-one correspondence between every term of that expansion and the particular graph on the $m \times L$ lattice ($N = m \cdot L$) can be established in the standard way (Blöte and Nightingale 1982).

Now, the partition function becomes

$$Z_{N,L} = q^N \cdot \sum_G u_1^{b_1(G)} \cdot u_2^{b_2(G)} \cdot \ldots \cdot u_L^{b_L(G)} \cdot q^{l(G)}. \tag{21}$$

The above summation goes over all possible graphs on the lattice with L different types of bonds described by $u_j = v_j/q$, $b_j(G)$ is the number of bonds of type j, and $l(G)$ is the number of independent loops on the graph G.

A certain number of different bonds in every single graph produce connections between sites. If one considers a strip of height L, then all possible interactions between all sites in that strip produce a particular state of connectivity α of the sites in the last column. The LR interactions produce a larger number of connectivities than in the 2D SR models.

The connectivity is represented by an integer, and this will be used as an index of the transfer matrix. The partition function Z_N can be written as a vector with components $Z_N(\alpha)$, where each component $Z_N(\alpha)$ describes the contribution to the partition function of all graphs that produce one particular state of connectivity α in the last column. By adding a new column, we get Z_{N+L} which again can be written as a vector $Z_{N+L}(\beta)$. The connection between those two partition functions is given by the transfer matrix in the following way

$$Z_{N+L}(\beta) = \sum_\alpha T(\beta, \alpha) Z_N(\alpha) \tag{22}$$

where the summation is over all graphs, produced by the added column, that lead from connectivity α to β. A transfer matrix decomposition analogous to that of section

2.1, leads to

$$T = qT_1 \cdot \ldots \cdot qT_L = qT_L = (qT_1 V^T)^L = \tilde{T}^L \tag{23}$$

where T_j describes the addition of the jth site to the column, and V is the matrix of the translation operator in the vertical strip direction. Further decomposition of T_1 is possible

$$T_1 = T_{1,L}(u_L) \cdot \prod_{j=L-1}^{1} T_{1,j}(u_j). \tag{24}$$

Each of the $T_{1,j}(u_j)$ matrices describes the presence of a u_j-type bond on the first site and has at most two non-zero elements per row. $T_{1,L}(u_L)$ is an upper triangular matrix, while the other matrices are lower triangular. It is easy to construct a matrix M which relates the lower triangular matrices

$$M^T T_{1,j}(x)M = T_{1,j+1}(x). \tag{25}$$

The matrix $\tilde{T}$ then becomes

$$\tilde{T} = qT_{1,L}(u_L) \prod_{j=L-1}^{1} (T_{1,L-1}M)_{u_j} V^T. \tag{26}$$

By using these decompositions, it is sufficient to keep in computer memory only four matrices: $T_{1,L}$, $T_{1,L-1}$, M and V, with at most two non-zero elements per row, which significantly reduces the amount of data to be stored.

Since we want to get information about LR correlations, we should also take care of the connectivity between sites in the last column and sites in the first column. In order to do so, one can imagine the top site in the last column of Z_N to be some kind of 'ghost' site representing the connection between the first site of the last column of Z_{N+L} and the sites of the first column. Thus we need a column of height L to describe all connectivities in the model with the range of interaction equal to $L-1$. In order to keep the 'ghost' site fixed, the matrix of the translation operator V will be constructed by acting with operator V on non-ghost sites only. Consequently, if we perform the following set of transformations

$$u_L \to u_{L-1}, u_{L-1} \to u_{L-2}, \ldots, u_2 \to u_1, u_1 \to 0 \tag{27}$$

we will obtain $\tilde{T}$ for the model with range $L-1$, with LR correlations included. With this redefinition, $\tilde{T}$ has the following structure

$$\tilde{T} = \begin{pmatrix} \tilde{T}_- & X \\ 0 & \tilde{T}_+ \end{pmatrix}$$

where $\tilde{T}_+$ and $\tilde{T}_-$ contain the largest $\mu_{1,L}$ and the second largest $\mu_{2,L}$ eigenvalue of the whole matrix $\tilde{T}$, respectively. The correlation length is given by the relation (16) again.

The order of $\tilde{T}$ for a model with range L is given by N_L, and the order of $\tilde{T}_+$ is given by N_{L-1}, where

$$N_L = \sum_{i=1}^{L+1} \varphi_i^L \tag{28}$$

$$\varphi_i^L = i \cdot \varphi_i^{L-1} + \varphi_{i-1}^{L-1} \tag{29}$$

for $L \geq 2$, and $\varphi_2^2 = 3$, $\varphi_1^n = \varphi_{n+1}^n = 1$. Some values of N_L are presented in table 1.

Table 1. N_L is the order of transfer matrix for continuous-q model with maximum range equal L.

L	2	3	4	5	6	7	8	9
N_L	5	15	52	203	877	4140	21147	115975

The present calculations were performed for the maximum range $L = 8$ with $q = \frac{1}{16}$, $\frac{1}{2}$, 1, 8, 16, 32 and 64. The largest eigenvalues of the matrices $\tilde{T}_{\pm}$, are extracted by using the method explained earlier (section 2.2) with similar accuracy and number of iterations.

3. Results

3.1. Convergence and extrapolation procedures

The critical temperature and exponent ν, given by the equations (7) and (8), still depend on the range M and it is important to apply the right extrapolation procedure. A simple analysis done within FSS (Privman and Fisher 1983) can be reproduced in the case of the FRS (Glumac and Uzelac 1989) and shows that the convergence is dominantly governed by the leading irrelevant field and the corresponding critical exponent $y_3 < 0$. On the basis of such analysis, we expect power-law convergence of T_c and ν in the large-M limit

$$T_{c,M} = T_c + a \cdot M^{y_3 - 1/\nu} \tag{30}$$

$$\nu_M^{-1} = \nu^{-1} + b(T_{c,M} - T_c) \cdot M^{1/\nu} + c \cdot M^{y_3} = \nu^{-1} + b' \cdot M^{y_3} \tag{31}$$

where a, b, b' and c are constants. Thus, we chose to fit the obtained results for $K_{c,M}$ and ν_M to the form

$$y_M = y_e + A \cdot M^{-x_y} \tag{32}$$

in the least-squares approximation (LSA), where y_e denotes the extrapolated quantity. The calculations are performed by taking the five largest values of M.

It should be noticed, however, that the Privman and Fisher (1983) analysis does not apply to the MF region and, consequently, the extrapolation form (32) becomes arbitrary there. On the other hand, in our previous analysis for the Ising model (Glumac and Uzelac 1989) it turns out that the extrapolation form

$$y_M = B + A\left(\frac{M-1}{M}\right)^{x_y} \tag{33}$$

gives better agreement with the exactly known results for ν in the MF region than the function of the form (32).

The error bars of the above extrapolations can be estimated only roughly, by looking for the remaining L-dependence of the results. Tables (2) and (3) present results in which the last digit is modified under the change of maximal range from $L-1$ to L. More careful examination of the errors shows that the size of the error estimate changes with q and σ, being less then 10% over most of the q and σ region. An exception is the small q and σ region, where the error is estimated to be a few times larger.

Table 2. (*a*) The extrapolated values of inverse critical temperature as a function of q and σ. For the error bars see text, (*b*) the critical temperature convergence exponent x_K as a function of q and σ.

σ/q	$\frac{1}{16}$	$\frac{1}{2}$	1	2	3	4	5	8	16	32	64
0.1	0.004	0.031	0.061	0.0927	0.136	0.203	0.28	0.48	0.78	1.00	1.14
0.2	0.007	0.052	0.102	0.1831	0.2701	0.362	0.45	0.64	0.91	1.12	1.31
0.3	0.010	0.077	0.149	0.2717	0.3862	0.489	0.576	0.76	1.02	1.24	1.45
0.4	0.014	0.108	0.204	0.3625	0.4939	0.601	0.690	0.875	1.14	1.37	1.59
0.5	0.020	0.147	0.270	0.4590	0.6013	0.713	0.803	0.99	1.26	1.53	1.75
0.6	0.028	0.198	0.351	0.5644	0.7143	0.829	0.920	1.12	1.40	1.69	1.94
0.7	0.041	0.269	0.452	0.6833	0.8374	0.954	1.046	1.24	1.54	1.84	2.15
0.8	0.064	0.375	0.584	0.8231	0.9774	1.093	1.185	1.384	1.69	2.00	2.336
0.9	0.121	0.552	0.763	0.9973	1.144	1.255	1.343	1.540	1.84	2.173	2.519
1.0	0.430	0.815	0.990	1.230	1.348	1.440	1.518	1.697	1.997	2.325	2.677

σ/q	$\frac{1}{16}$	$\frac{1}{2}$	1	2	3	4	5	8	16	32	64
0.1	1.5	1.4	1.3	0.86	0.81	0.8	0.9	1.0	1.1	1.0	0.8
0.2	1.5	1.4	1.3	1.02	0.99	1.1	1.1	1.2	1.2	1.0	0.9
0.3	1.6	1.5	1.4	1.13	1.13	1.2	1.3	1.3	1.2	1.0	0.9
0.4	1.6	1.5	1.4	1.19	1.21	1.3	1.3	1.3	1.2	1.0	0.9
0.5	1.6	1.4	1.3	1.22	1.25	1.3	1.4	1.3	1.2	1.1	0.9
0.6	1.6	1.4	1.3	1.22	1.26	1.3	1.4	1.4	1.3	1.2	1.0
0.7	1.5	1.3	1.3	1.20	1.26	1.3	1.4	1.4	1.4	1.3	1.2
0.8	1.4	1.2	1.3	1.22	1.31	1.4	1.5	1.5	1.5	1.4	1.4
0.9	1.1	1.5	1.7	1.88	1.70	1.7	1.7	1.7	1.6	1.6	1.6
1.0	1	1	1	1	1	1	1	1	1	1	1

3.2. Critical temperature

The inverse critical temperature $K_{c,M}$, defined by equation (7), is calculated with an accuracy higher than 10^{-10} for different values of Potts states and with σ as a parameter.

When increasing the exponent of LR interaction, there will be a value of σ beyond which the SR critical behaviour (with $T_c = 0$) takes place. For the Ising model ($q = 2$) it was anlytically shown (Dyson 1969) that this σ is equal to 1. As argued more generally (Fisher *et al* 1972, Sak 1973) this exchange of regimes should occur when the correlation function exponent η of the LR system becomes smaller than or equal to the SR one, η_{SR}. Thus, in the present case we expect that $\sigma_c = 1$ for all values of q. This expectation is confirmed by our numerical results. In earlier work (Glumac and Uzelac 1989, 1991) we have been able to detect the appearance of SR forces governing critical behaviour by the change of $K_{c,M}$ from descending to ascending sequence, and by the sudden increase of convergence exponent. In the present calculations, both of these phenomena were observed for any q considered at $\sigma \approx 1$. The extrapolated values for the critical temperature are presented in figures 1(*a*) and 1(*b*).

We have used the LSA extrapolation method with the form (32) to calculate $K_{c,e}$ and the leading convergence exponent x_K as a function of q and σ which are presented in tables 2(*a*) and 2(*b*), respectively. The only point where this could not be applied is $\sigma = 1$, where the simple convergence expression does not apply any more. In order to avoid non-monotonic behaviour, we have used only the data for the three largest

Table 3. (a) The extrapolated values of ν^{-1} as a function of q and σ. For the error bars see text; (b) the convergence exponent x_ν as a function of q and σ.

σ/q	$\frac{1}{16}$	$\frac{1}{2}$	1	2	3	4	5	8	16	32	64
0.1	0.11	0.12	0.12	0.101	0.091	0.12	0.16	0.21	0.31	—	—
0.2	0.18	0.20	0.21	0.202	0.215	0.24	0.30	0.45	0.75	—	—
0.3	0.32	0.31	0.31	0.301	0.323	0.41	0.53	1.1	4.1	—	—
0.4	0.34	0.34	0.35	0.373	0.481	0.59	0.89	3.0	—	—	—
0.5	0.33	0.34	0.37	0.430	0.577	0.77	1.2	5.0	—	—	—
0.6	0.30	0.32	0.37	0.501	0.664	0.83	1.2	2.3	—	—	—
0.7	0.25	0.29	0.35	0.518	0.636	0.78	1.0	1.4	4.6	—	—
0.8	0.18	0.24	0.32	0.483	0.574	0.67	0.80	0.99	1.3	1.4	—
0.9	0.10	0.21	0.30	0.405	0.491	0.56	0.62	0.76	0.97	1.1	—
1.0	0.05	0.18	0.25	0.309	0.393	0.46	0.52	0.63	0.79	0.92	1.0

σ/q	$\frac{1}{16}$	$\frac{1}{2}$	1	2	3	4	5	8	16	32	64
0.1	—	—	—	—	0.94	0.89	0.84	0.83	0.61	—	—
0.2	—	—	—	—	0.75	0.84	0.76	0.56	0.24	—	—
0.3	0.72	0.74	0.75	—	0.78	0.65	0.50	0.24	0.04	—	—
0.4	0.71	0.72	0.72	—	0.52	0.48	0.30	0.08	—	—	—
0.5	0.72	0.72	0.71	0.70	0.48	0.37	0.23	0.05	—	—	—
0.6	0.75	0.74	0.72	0.49	0.40	0.36	0.24	0.12	—	—	—
0.7	0.78	0.77	0.77	0.43	0.46	0.41	0.30	0.24	0.06	—	—
0.8	0.81	0.92	0.99	0.43	0.59	0.57	0.47	0.42	0.31	0.32	—
0.9	0.87	1.00	1.00	1.41	2.16	1.32	1.08	0.84	0.62	0.62	—
1.0	1	1	1	1	1	1	1	1	1	1	1

values of M fitting a linear relation to them ($x_K = 1$). The error should not be important, since for $\sigma = 1$ the variation of data with M is rather weak.

3.3. Critical exponent ν

The correlation length critical exponents ν_M have been calculated using equation (8). The accuracy of results is reduced (with respect to that of the critical temperature) to the order of 10^{-6} due to the numerical differentiation.

Following the form (31) for the scaling correction in non-MF region, one can expect that the correction terms to ν_M^{-1} would be smaller when calculating the ν_M on the temperature $K_{c,e}$ instead of $K_{c,M}$. This is indeed the case for the $q=2$ ($s=\frac{1}{2}$ Ising) model, where detailed comparison between $\nu_M(K_{c,M})$ and $\nu_M(K_{c,e})$ has been given (Glumac and Uzelac 1989). In the present model we observe better convergence of the data calculated from $K_{c,e}$ in the LR σ-region, for all q.

According to the discussion at the end of section 3.1, the extrapolations in the low σ region where MF behaviour is expected, were performed at the temperature $K_{c,M}$ using the extrapolation function of the form (33). Since the σ_{MF} border is still an open question for $q>2$ (see later in text), we have used equation (33) for $\sigma<0.3$, $q\leqslant 1$, and for $\sigma<0.5$, $q=2$.

The extrapolated results are presented in figure 2. Tables 3(a) and 3(b) show results for ν_e^{-1} and the corresponding convergence exponents respectively. Similarly to pre-
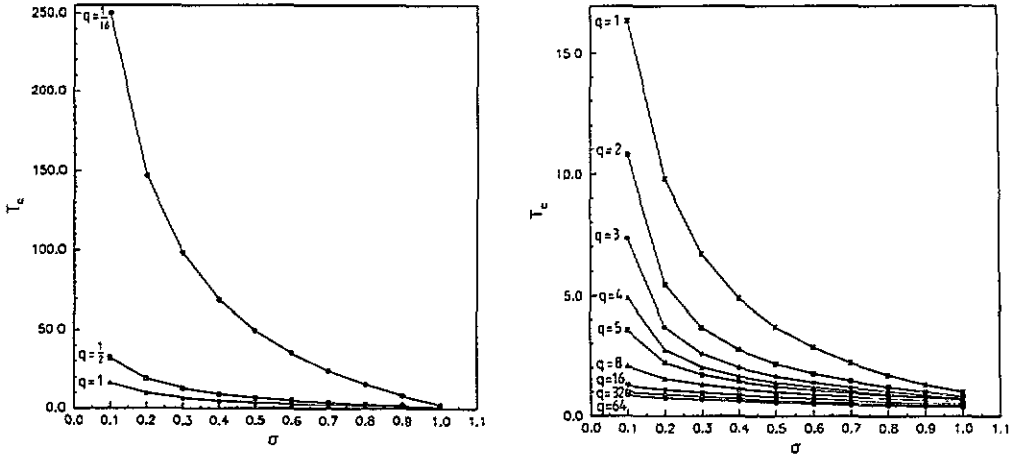
Figure 1. (a) The extrapolated critical temperature as a function of σ, (b) the extrapolated critical temperature as a function of σ. Figures (a) and (b) have common $q = 1$ line which allows the comparison of scales on both figures.
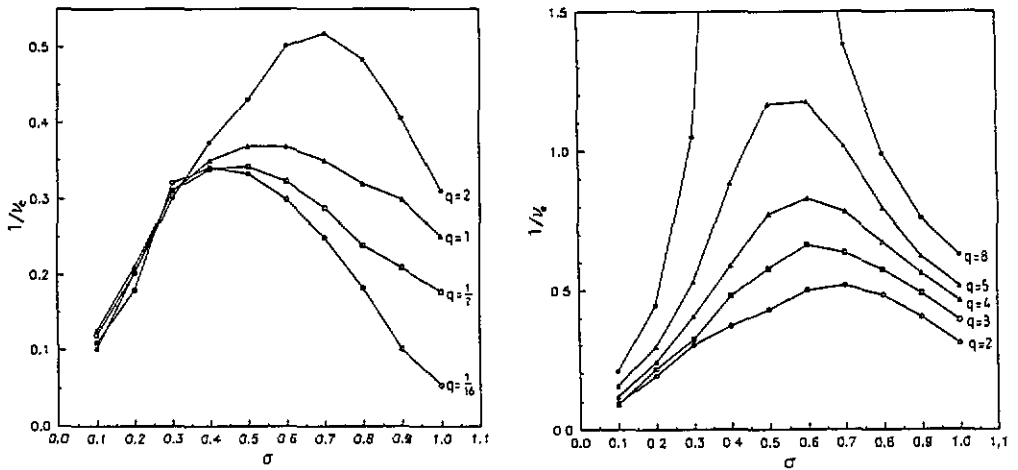

Figure 2. (a) The exponent ν_c^{-1} as a function of σ; (b) the exponent ν_c^{-1} as a function of σ. Figures (a) and (b) have common $q = 2$ line which allows the comparison of scales on both figures.

ceding section, for $\sigma = 1$ the LSA procedure was performed by imposing $x_\nu = 1$ in equation (32).

The special case $q = 2$ which corresponds to the Ising model was extensively studied in our earlier work (Glumac and Uzelac 1989) with a maximum range equal 10. The present work permits the range to be extended to 20. Similar values of ν_e were obtained in both works, which suggests that an increase of range from 10 to 20 does not change the accuracy significantly.

In table 4, our results are compared to the ε-expansion results of Theumann and Gusmão (1985). A difference Δ of only few per cent is obtained. Figure 2(a) suggests $\sigma = \frac{1}{3}$ as the MF border and a q-independent value of $\nu_{MF}^{-1} = \sigma$ for all $q \leqslant 1$.

Table 4. The comparison between extrapolated values ν_c^{-1} and Theumann and Gusmão values ν_{TG}^{-1} for $\sigma = 0.4$.

q	ν_c^{-1}	ν_{TG}^{-1}	Δ (%)
$\frac{1}{16}$	0.340	0.3250	4.5
$\frac{1}{2}$	0.338	0.3300	2.5
1	0.349	0.3387	3
2	0.373	0.4	7

We have not found any analytical or numerical estimate for the values of σ_{MF} or ν for $q \geqslant 3$. In that situation we decided to calculate ν at the temperature $K_{c,e}$ since the scaling law $\xi_M/M = \text{constant}$ is more accurate at that temperature than at $K_{c,M}$. Also, we decided to extrapolate using function (32) which possesses a more transparent L-dependence.

For large q ($q \geqslant 8$), only the continuous formalism could be applied, which does not permit going beyond the range $L = 8$. In contrast to the small q case, for large q this range was not sufficient to find a good extrapolation. Namely, when applying the LSA there, with the assumed form (31), one obtains $x_\nu \to 0$ so that the correction terms become of the same order as the leading term and ν^{-1} diverges. This behaviour of ν suggests a change of regime of convergence and brings up the question of the order of transition.

For all values of q the exponent ν increases as $\sigma \to 1$, which in the case $q = 2$ appeared (Glumac and Uzelac 1989) as an indication of the essential singularity in this limit (Kosterlitz 1976).

4. Conclusion and discussion

By using a FRS method combined with transfer matrix calculations we have been able to study the long-range Potts model on the one-dimensional lattice in a wide range of values for the number of states q and the interaction exponent σ. We have shown that for this problem the transfer matrix can be decomposed into matrices of much simpler form. A different transfer matrix procedure can be done in the case of continuous q, so that our analysis could be extended to some non-integer q cases of interest.

The study is concentrated on the phase diagram and the correlation length critical exponent ν.

The critical temperature shows monotonic decrease with q and σ and has a finite value at $\sigma = 1$. The change of behaviour of $K_{c,M}$ at $\sigma \approx 1$ indicates the exchange from LR to SR regime, established for any q. The critical exponent was calculated at the extrapolated critical temperature, and good agreement with values obtained by other authors was observed. Generally, a good accuracy is harder to obtain in the small-σ region due to the very long-range of interaction and in the $\sigma \to 1$ region where the range of interaction ceases to be a good scaling variable.

There are two points that require some further discussion.

First is the question of mean-field border. The present method, unlike the FSS, has the advantage that it can be applied within the mean-field region, but on the other hand, passes smoothly between the two regions (Glumac and Uzelac 1989, 1991),

which does not permit detection of the mean-field border σ_{MF}. Another way to detect this border could be numerical, by observing the change of behaviour of ν as a function of σ. But our previous results on the Ising model where σ_{MF} is known, show that close to it the numerical results for ν are not sharp enough to locate this point with precision. In the present case, we can only confirm that our results are consistent with σ_{MF} as known from the literature for a few particular cases: $q = 2$ ($\sigma_{MF} = \frac{1}{2}$; Fisher *et al* 1972), $q = 1$ ($\sigma_{MF} = \frac{1}{3}$; Priest and Lubensky 1976), $q = \frac{1}{2}$ ($\sigma_{MF} = \frac{1}{3}$; Aharony 1978, Aharony and Pfeuty 1979), but the precise location of σ_{MF} for arbitrary q is left unestimated.

A second point is the possible appearance of first-order transition for some $q > q_c$. Within the transfer matrix formalism this problem was considered for the 2D SR case where q_c exists and is exactly known ($q_c = 4$; Baxter 1973). Iglói and Sólyom (1983) have shown that the first-order transition is connected with the crossing of the largest and the third largest eigenvalue of the transfer matrix for finite length chain and large q. We have applied a similar calculation in our model considering two groups of parameters ($q = 100$, $\sigma = 0.8$, $L = 4$ and $q = 300$, $\sigma = 0.4$, $L = 4$). Both cases give a negative result; the first and third eigenvalues do not intersect as a function of temperature. By analogy with Iglói and Sólyom, that result can be interpreted as the absence of a first-order transitions for any q in the 1D LR Potts model. Since this is opposite to the indications of the behaviour of ν for large q (at the end of section 3.3), the question of the order of transition for large q is still left open.

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